CH 8.
HEAPS AND PRIORITY QUEUES

ACKNOWLEDGEMENT: THESE SLIDES ARE ADAPTED FROM SLIDES PROVIDED WITH DATA STRUCTURES AND ALGORITHMS IN C++, GOODRICH, TAMASSIA AND MOUNT (WILEY 2004) AND SLIDES FROM NANCY M. AMATO

OUTLINE AND READING

- PriorityQueue ADT (Ch. 8.1)
  - Total order relation (Ch. 8.1.1)
  - Comparator ADT (Ch. 8.1.2)
  - Sorting with a Priority Queue (Ch. 8.1.5)
- Implementing a PQ with a list (Ch. 8.2)
  - Selection-sort and Insertion Sort (Ch. 8.2.2)
- Heaps (Ch. 8.3)
  - Complete Binary Trees (Ch. 8.3.2)
  - Implementing a PQ with a heap (Ch. 8.3.3)
  - Heapsort (Ch. 8.3.5)

PRIORITY QUEUES

- Stores a collection of elements each with an associated “key” value
  - Can insert as many elements in any order
  - Only can inspect and remove a single element – the minimum (or maximum depending) element
- Applications
  - Standby Flyers
  - Auctions
  - Stock market

TOTAL ORDER RELATION

- Keys in a priority queue can be arbitrary objects on which an order is defined, e.g., integers
- Two distinct items in a priority queue can have the same key

- Mathematical concept of total order relation $\leq$
  - Reflexive property:
    $$k \leq k$$
  - Antisymmetric property:
    if $\mathrm{k}_1 \leq k_2$ and $k_2 \leq k_1$, then $k_1 = k_2$
  - Transitive property:
    if $k_1 \leq k_2$ and $k_2 \leq k_3$ then $k_1 \leq k_3$

COMPARATOR ADT

- A *comparator* encapsulates the action of comparing two objects according to a given total order relation
- A generic priority queue uses a comparator as a template argument, to define the comparison function ($\leq$)
- The comparator is external to the keys being compared. Thus, the same objects can be sorted in different ways by using different comparators.
- When the priority queue needs to compare two keys, it uses its comparator

PRIORITY QUEUE ADT

- A priority queue stores a collection of items each with an associated "key" value
- Main methods
  - insert(e) – inserts an element e
  - removeMin() – removes the item with the smallest key
  - min() – return an element with the smallest key
  - size(), empty()

PRIORITYQUEUESORT()
SORTING WITH A PRIORITY QUEUE

- We can use a priority queue to sort a set of comparable elements
- Insert the elements one by one with a series of $\text{insert}(e)$ operations
- Remove the elements in sorted order with a series of $\text{removeMin}()$ operations
- Running time depends on the PQ implementation

<u>Algorithm *PriorityQueueSort*()</u>

Input: List L storing n elements and a Comparator C

Output: Sorted List L

1. Priority Queue P using comparator C
2. **while** $\neg L.\text{empty}()$ **do**
3. $\quad P.\text{insert}(L.\text{front}())$
4. $\quad L.\text{eraseFront}()$
5. **while** $\neg P.\text{empty}()$ **do**
6. $\quad L.\text{insertBack}(P.\text{min}())$
7. $\quad P.\text{removeMin}()$
8. **return** L

LIST-BASED PRIORITY QUEUE

Unsorted list implementation

- Store the items of the priority queue in a list, in arbitrary order

- Performance:
  - insert(e) takes $O(1)$ time since we can insert the item at the beginning or end of the list
  - removeMin() and min() take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

Sorted list implementation

- Store the items of the priority queue in a list, sorted by key

- Performance:
  - insert(e) takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin() and min() take $O(1)$ time since the smallest key is at the beginning of the list

SELECTION-SORT

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list

- Running time of Selection-sort:
  - Inserting the elements into the priority queue with n $\text{insert}(e)$ operations takes $O(n)$ time
  - Removing the elements in sorted order from the priority queue with n $\text{removeMin}()$ operations takes time proportional to

$$\sum_{i=0}^{n} n - i = n + (n-1) + \cdots + 2 + 1 = O(n^2)$$

- Selection-sort runs in $O(n^2)$ time

EXERCISE
SELECTION-SORT

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted list (do n $\text{insert}(e)$ and then n $\text{removeMin}()$)

- Illustrate the performance of selection-sort on the following input sequence:
  - (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)

INSERTION-SORT

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted List

- Running time of Insertion-sort:
  - Inserting the elements into the priority queue with n $\text{insert}(e)$ operations takes time proportional to

$$\sum_{i=0}^{n} i = 1 + 2 + \cdots + n = O(n^2)$$

  - Removing the elements in sorted order from the priority queue with a series of n $\text{removeMin}()$ operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

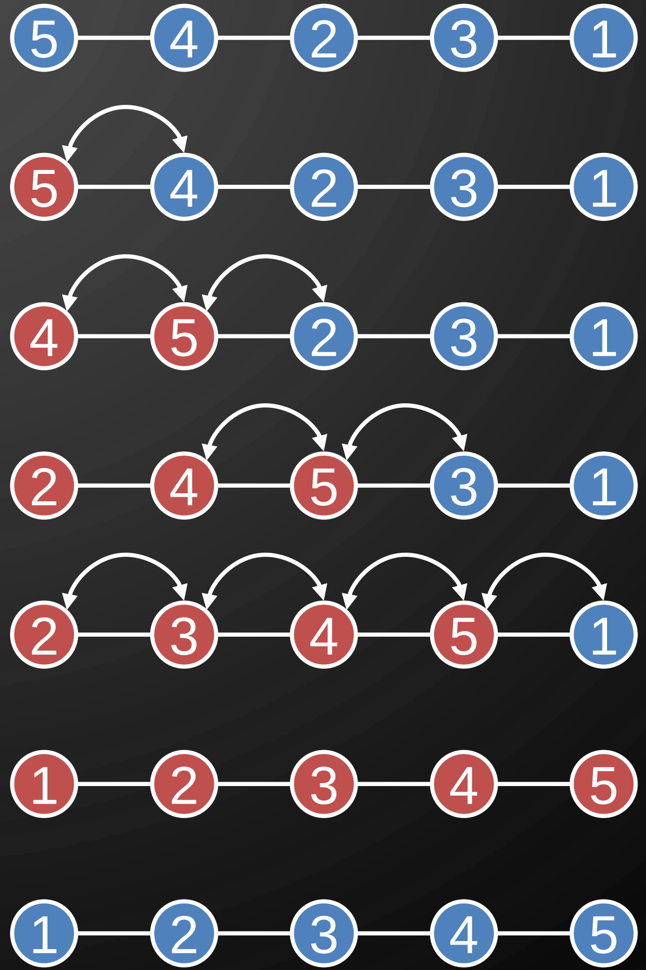
EXERCISE
INSERTION-SORT

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted list (do n $\text{insert}(e)$ and then n $\text{removeMin}()$)

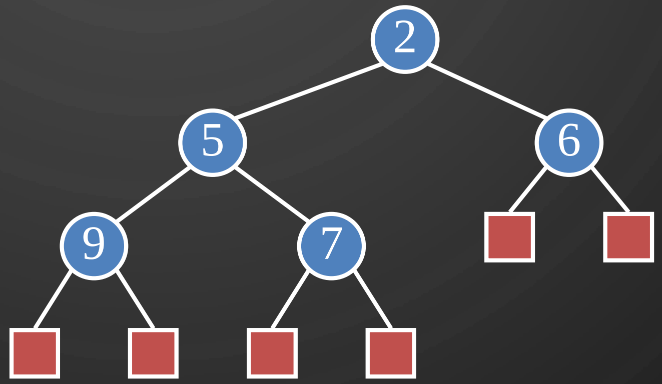
- Illustrate the performance of insertion-sort on the following input sequence:
  - (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)

IN-PLACE INSERTION-SORT

- Instead of using an external data structure, we can implement selection-sort and insertion-sort in-place (only O(1) extra storage)
- A portion of the input list itself serves as the priority queue
- For in-place insertion-sort
  - We keep sorted the initial portion of the list
  - We can use $\mathrm{swap}(i, j)$ instead of modifying the list

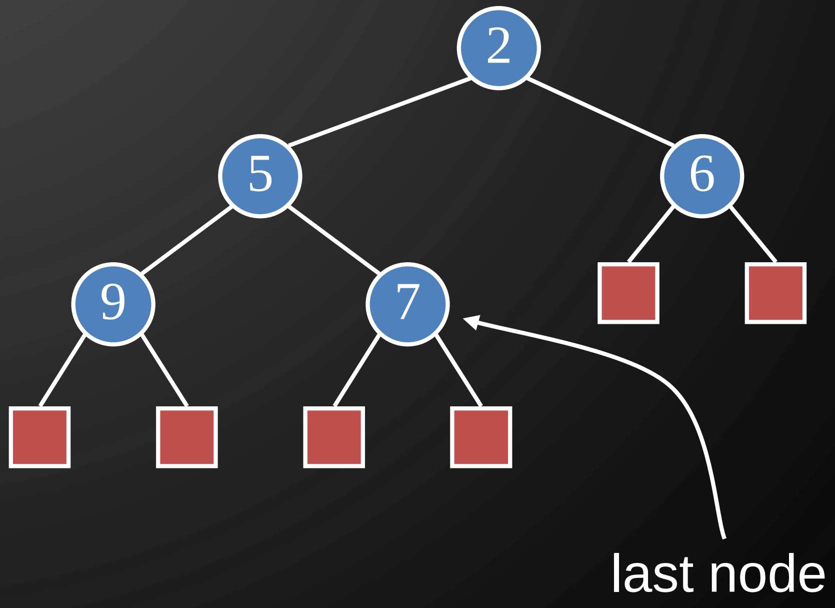


HEAPS



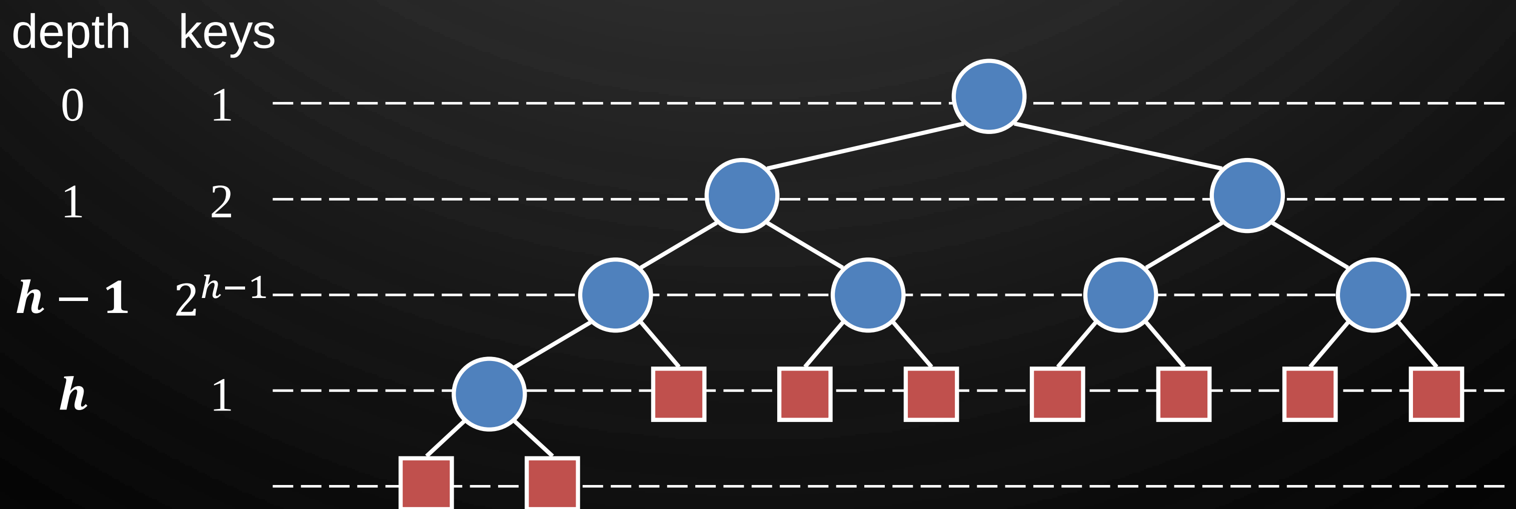
WHAT IS A HEAP?

- A heap is a binary tree storing keys at its internal nodes and satisfying the following properties:
  - Heap-Order: for every node v other than the root, $\text{key}(v) \geq \text{key}(v.\text{parent}())$
  - Complete Binary Tree: let h be the height of the heap
    - for $i = 0 \dots h-1$, there are 2^i nodes on level i
    - at level h, nodes are filled from left to right
- Can be used to store a priority queue efficiently



HEIGHT OF A HEAP

- Theorem: A heap storing n keys has height $O(\log n)$
- Proof: (we apply the complete binary tree property)
  - Let h be the height of a heap storing h keys
  - Since there are 2^i keys at level $i = 0 \ldots h-1$ and at least one key on level h, we have
  $$n \geq 1 + 2 + 4 + \cdots + 2^{h-1} + 1 = \left(2^h - 1\right) + 1 = 2^h$$
  - Level h has at most 2^h nodes: $n \leq 2^{h+1} - 1$
  - Thus, $\log(\mathrm{n}+1) - 1 \leq h \leq \log n$ ■

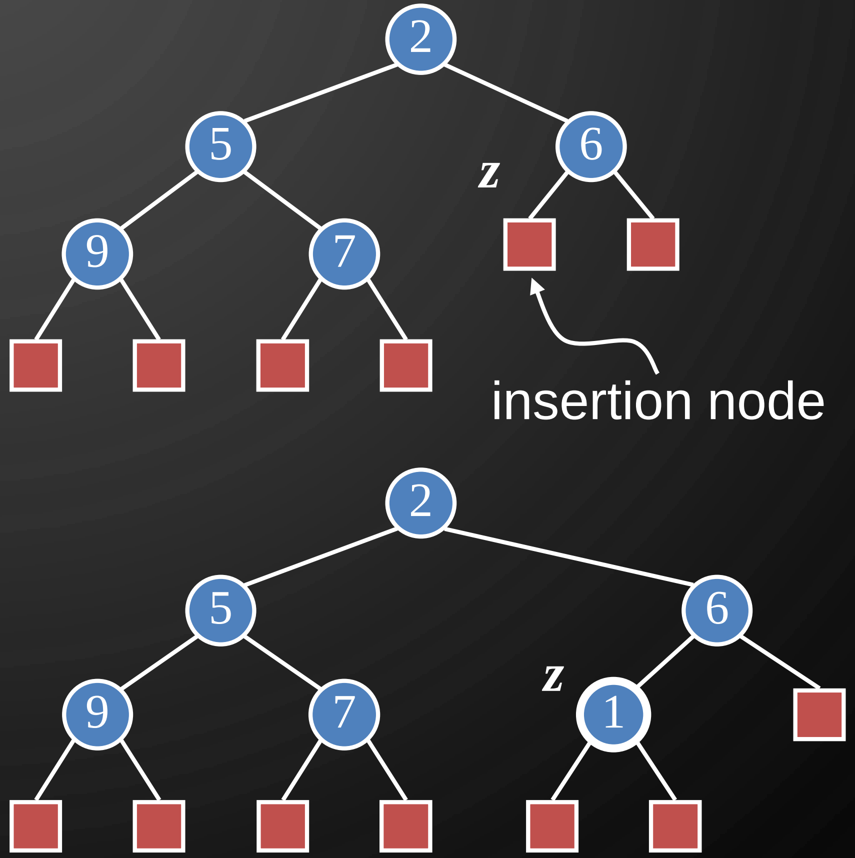


EXERCISE
HEAPS

- Let H be a heap with 7 distinct elements (1,2,3,4,5,6, and 7). Is it possible that a preorder traversal visits the elements in sorted order? What about an inorder traversal or a postorder traversal? In each case, either show such a heap or prove that none exists.

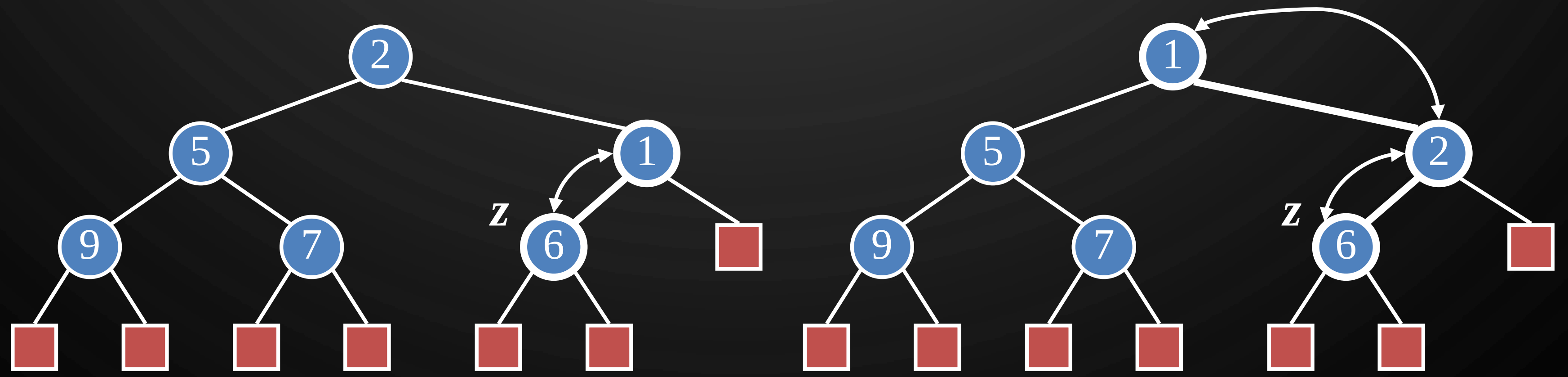
INSERTION INTO A HEAP

- insert(e) consists of three steps
  - Find the insertion node z (the new last node)
  - Store e at z and expand z into an internal node
  - Restore the heap-order property (discussed next)



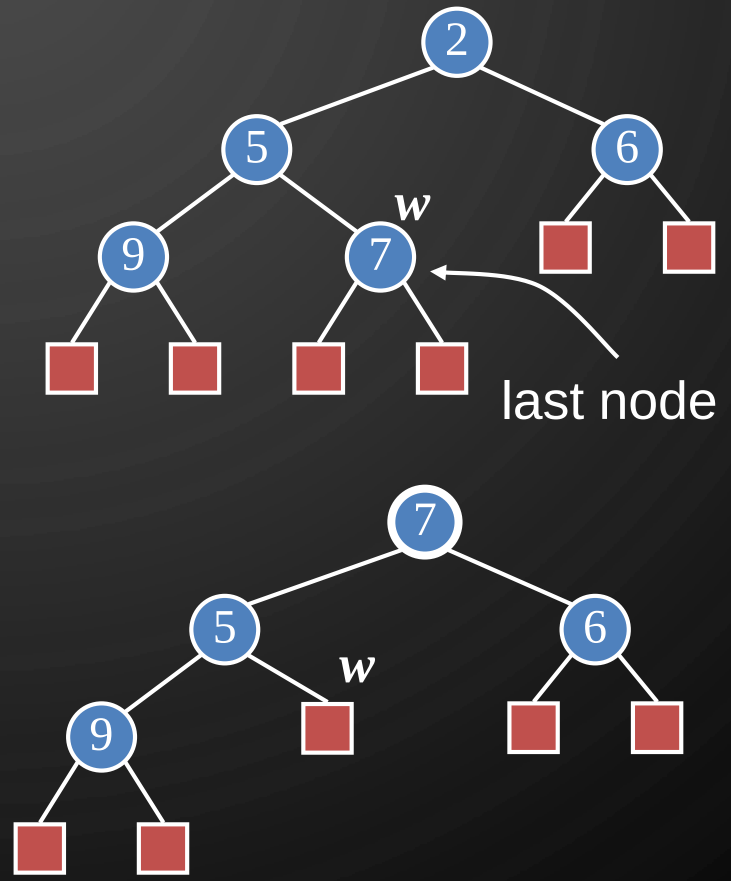
UPHEAP

- After the insertion of a new element e, the heap-order property may be violated
- Up-heap bubbling restores the heap-order property by swapping e along an upward path from the insertion node
- Upheap terminates when e reaches the root or a node whose parent has a key smaller than or equal to $\text{key}(e)$
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



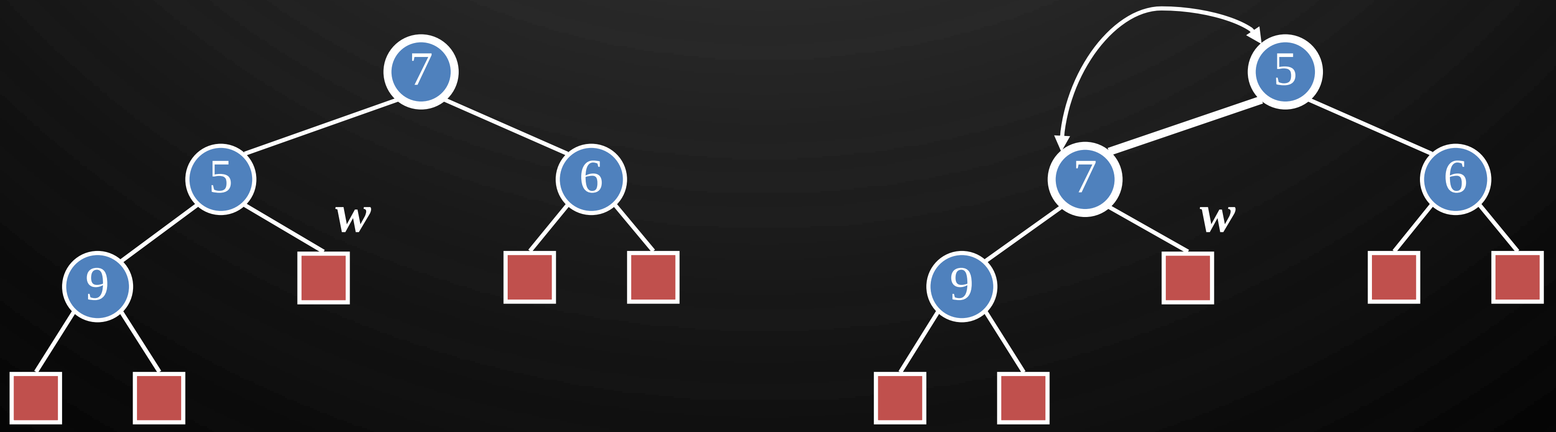
REMOVAL FROM A HEAP

- removeMin() corresponds to the removal of the root from the heap
- The removal algorithm consists of three steps
  - Replace the root with the element of the last node w
  - Compress w and its children into a leaf
  - Restore the heap-order property (discussed next)



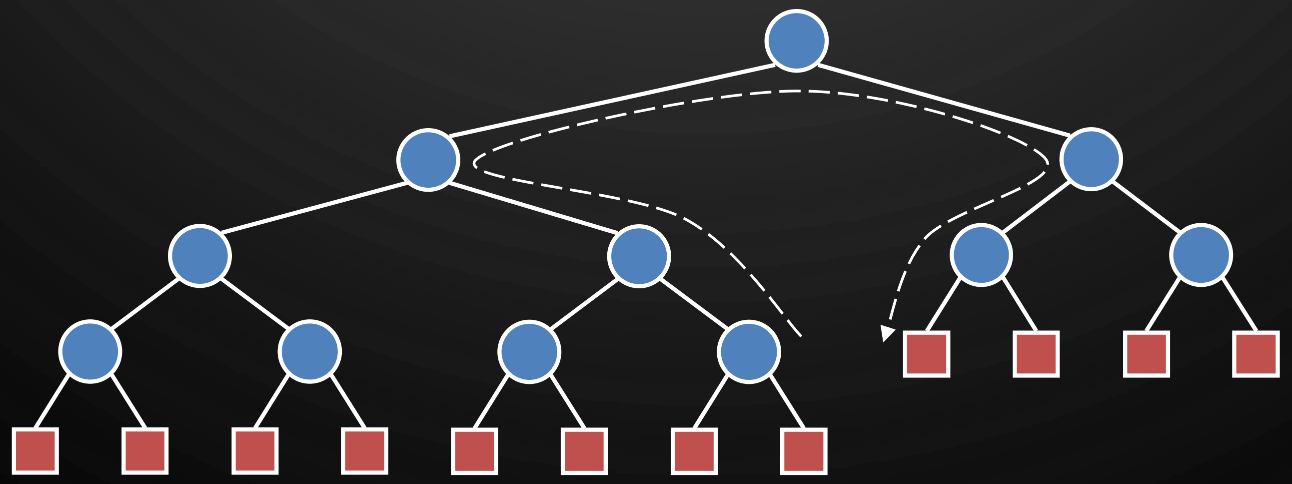
DOWNHEAP

- After replacing the root element of the last node, the heap-order property may be violated
- Down-heap bubbling restores the heap-order property by swapping element e along a downward path from the root
- Downheap terminates when e reaches a leaf or a node whose children have keys greater than or equal to $\text{key}(e)$
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time

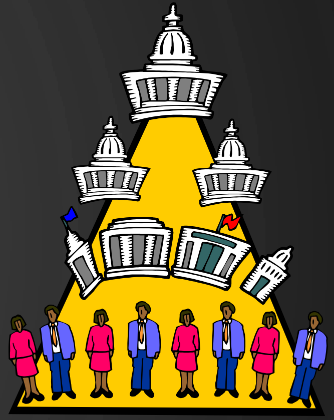


UPDATING THE LAST NODE

- The insertion node can be found by traversing a path of O(log n) nodes
  - Go up until a left child or the root is reached
  - If a left child is reached, go to the right child
  - Go down left until a leaf is reached
- Similar algorithm for updating the last node after a removal



HEAP-SORT



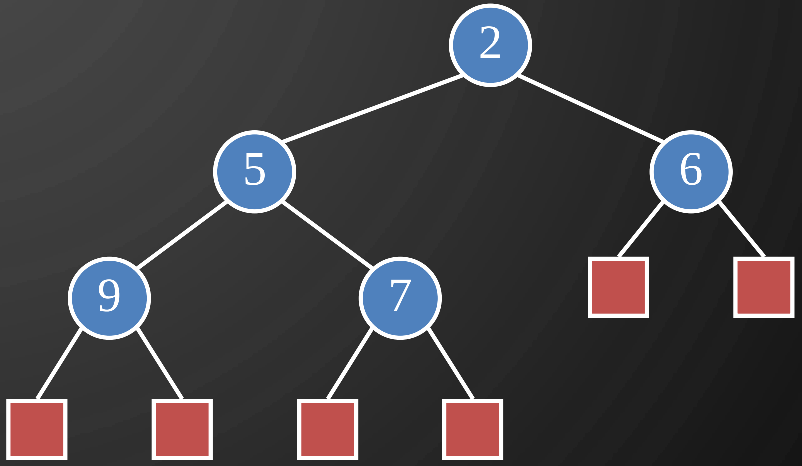
- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - $\text{insert}(e)$ and $\text{removeMin}()$ take $O(\log n)$ time
  - $\text{min}()$, $\text{size}()$, and $\text{empty}()$ take $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort

EXERCISE
HEAP-SORT

- Heap-sort is the variation of PQ-sort where the priority queue is implemented with a heap (do n $\text{insert}(e)$ and then n $\text{removeMin}()$)
- Illustrate the performance of heap-sort on the following input sequence (draw the heap at each step):
    - (22, 15, 36, 44, 10, 3, 9, 13, 29, 25)

VECTOR-BASED HEAP IMPLEMENTATION

- We can represent a heap with n elements by means of a vector of length $n+1$
  - Links between nodes are not explicitly stored
  - The leaves are not represented
  - The cell at index 0 is not used
- For the node at index i
  - the left child is at index $2i$
  - the right child is at index $2i+1$
- $\text{insert}(e)$ corresponds to inserting at index $n+1$
- $\text{removeMin}()$ corresponds to removing element at index n
- Yields in-place heap-sort



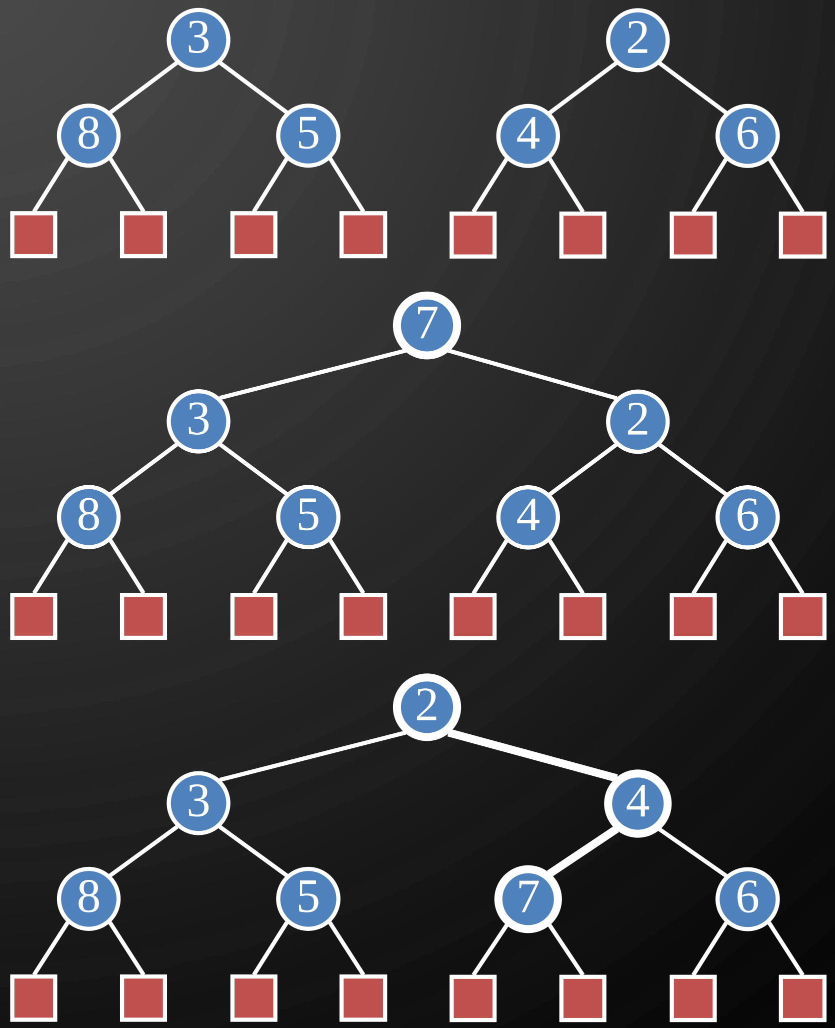
	2	5	6	9	7
0	1	2	3	4	5

PRIORITY QUEUE SUMMARY

	insert(e)	removeMin()	PQ-Sort total
Ordered List (Insertion Sort)	$O(n)$	$O(1)$	$O(n^2)$
Unordered List (Selection Sort)	$O(1)$	$O(n)$	$O(n^2)$
Binary Heap, Vector-based Heap (Heap Sort)	$O(\log n)$	$O(\log n)$	$O(n \log n)$

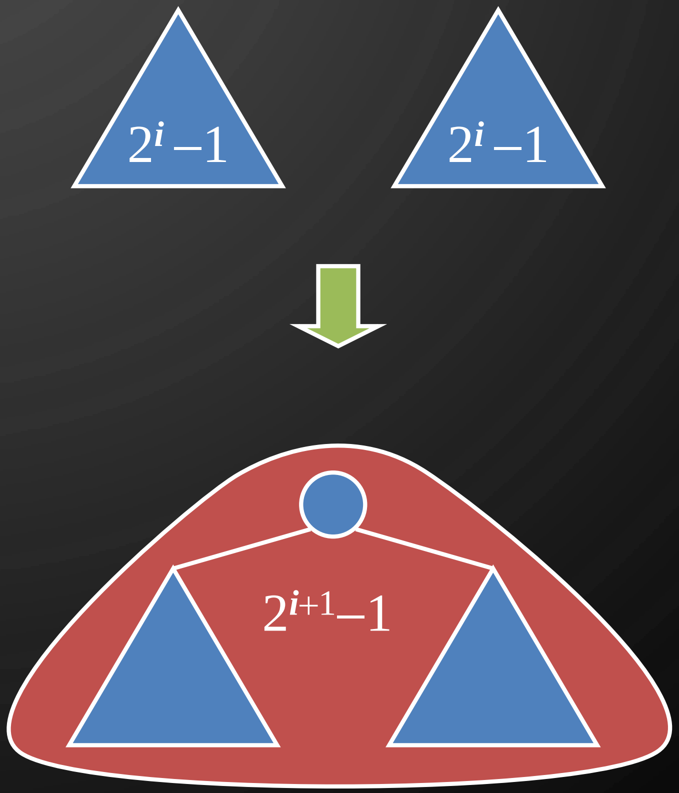
MERGING TWO HEAPS

- We are given two two heaps and a new element e
- We create a new heap with a root node storing e and with the two heaps as subtrees
- We perform downheap to restore the heap-order property

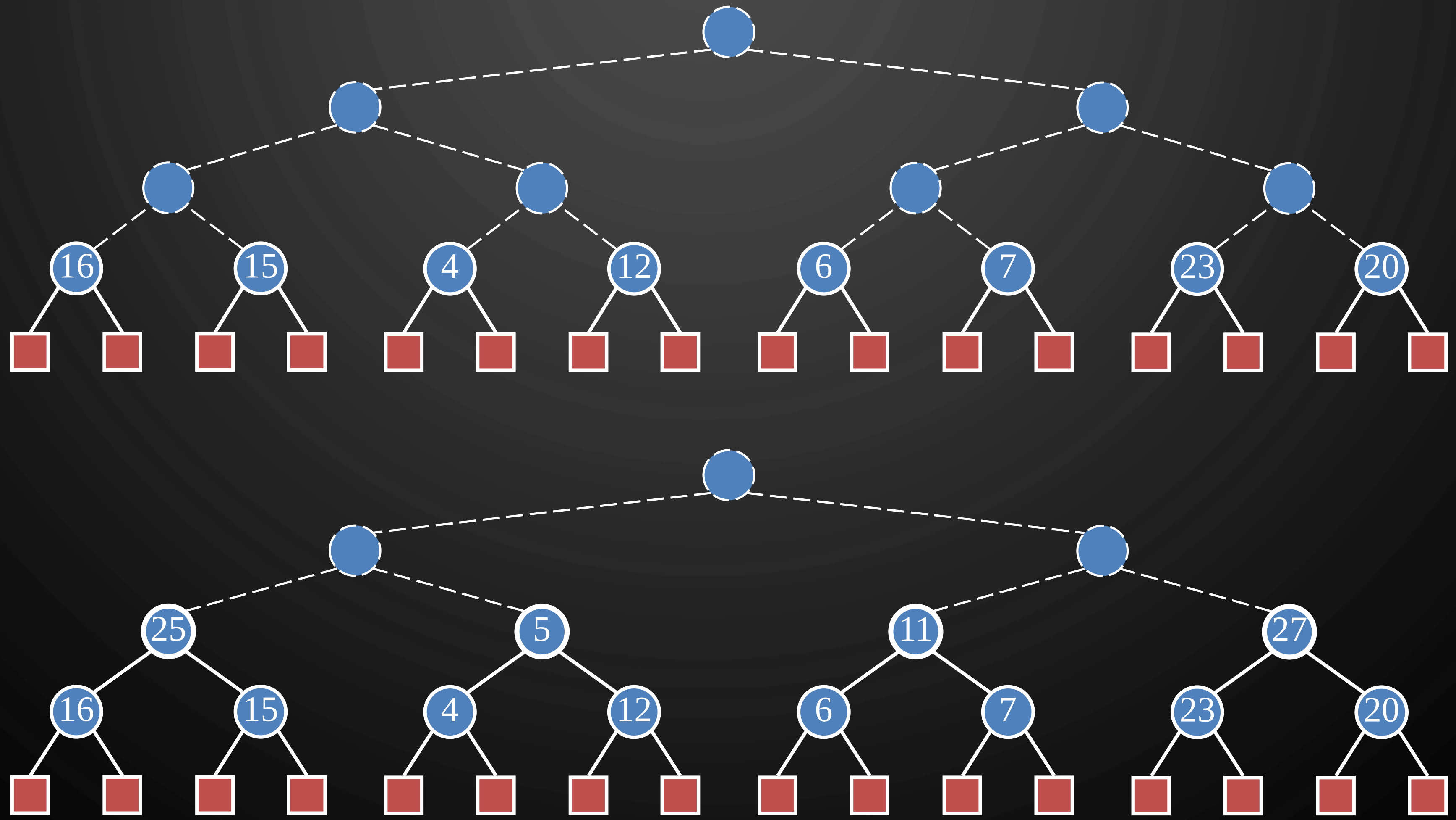


BOTTOM-UP HEAP CONSTRUCTION

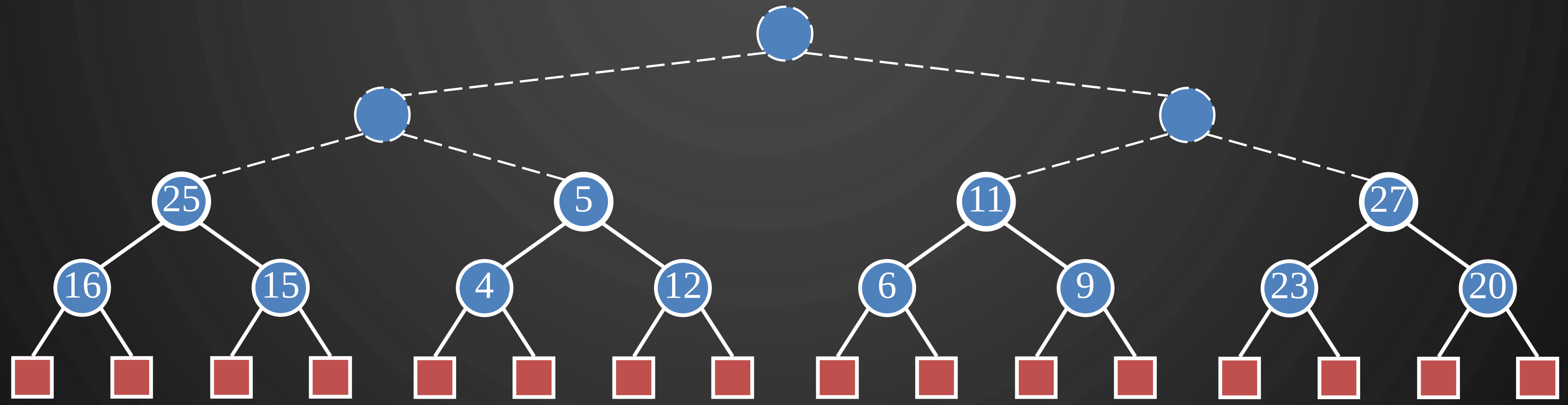
- We can construct a heap storing n given elements in using a bottom-up construction with $\log n$ phases
- In phase i, pairs of heaps with $2^i - 1$ elements are merged into heaps with $2^{i+1} - 1$ elements

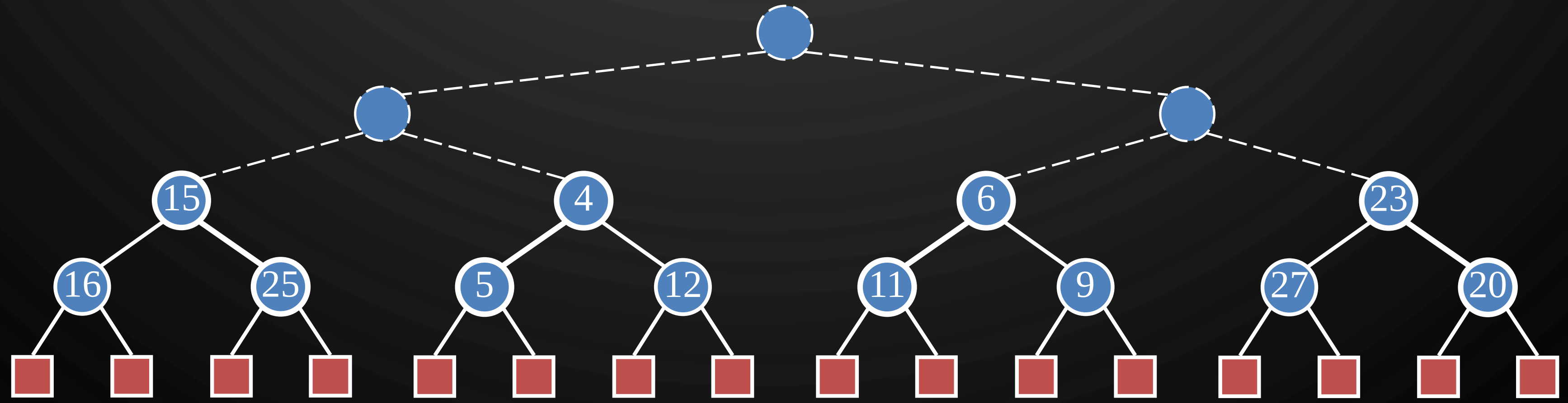


EXAMPLE

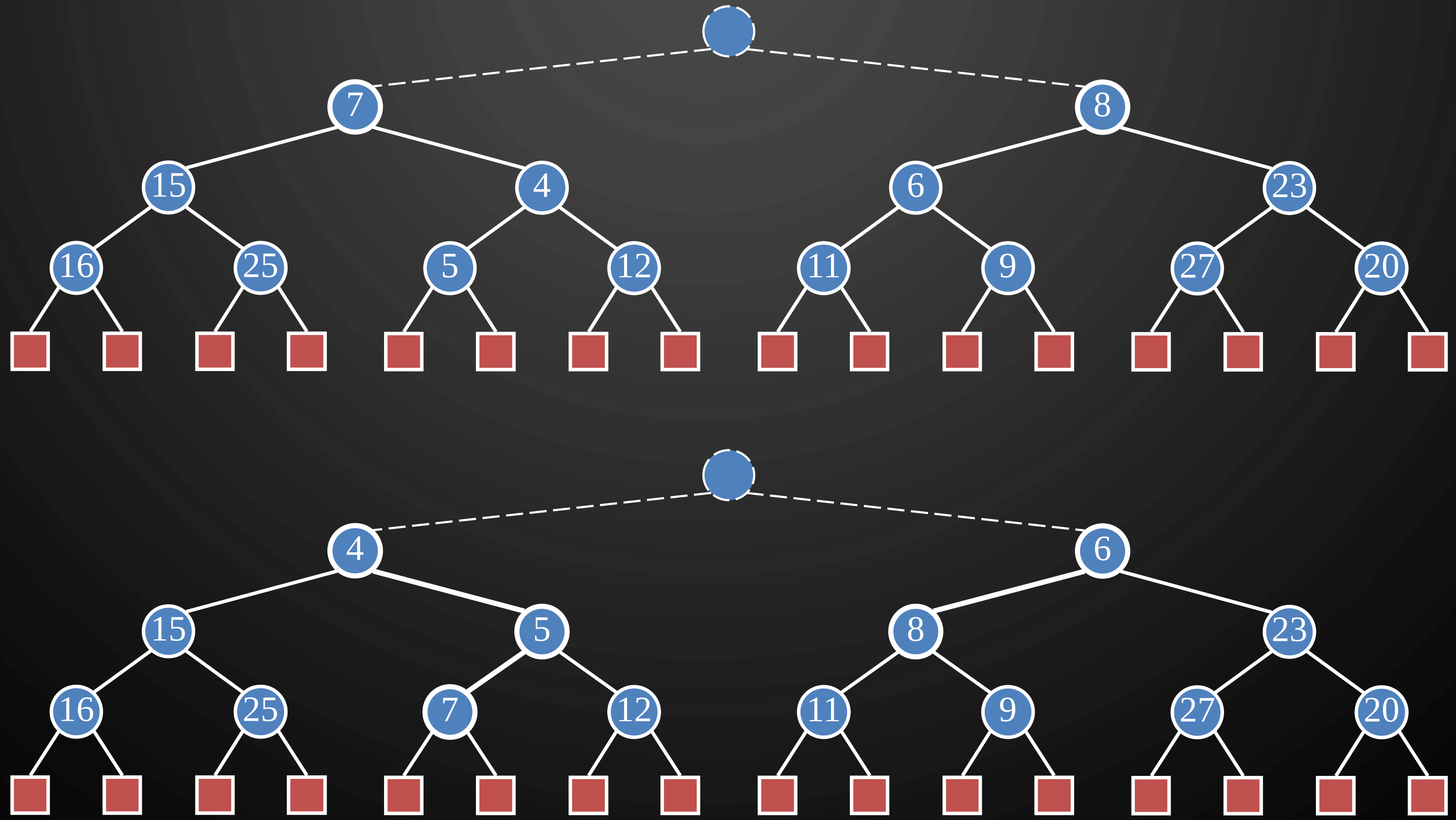


EXAMPLE

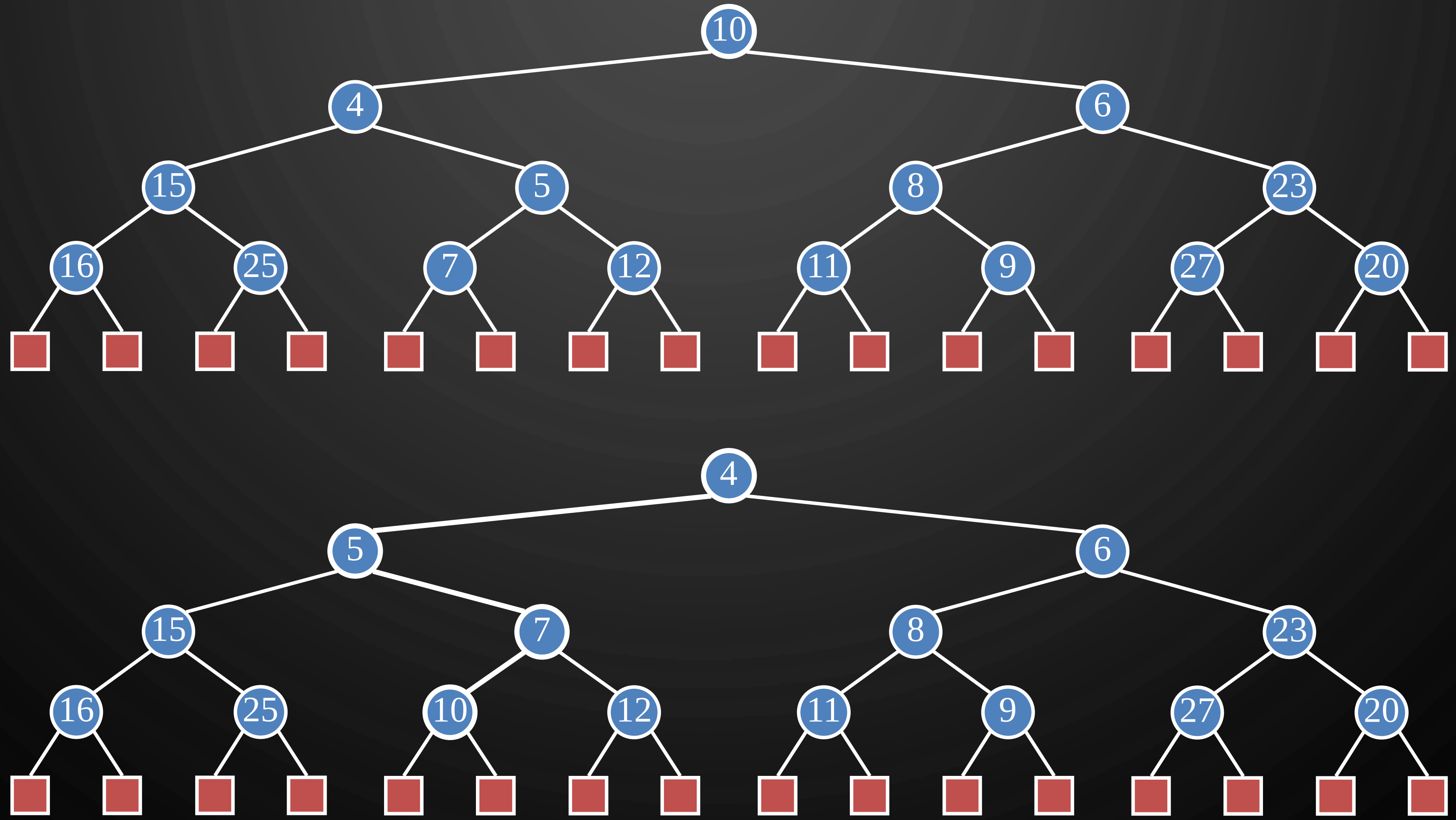




EXAMPLE

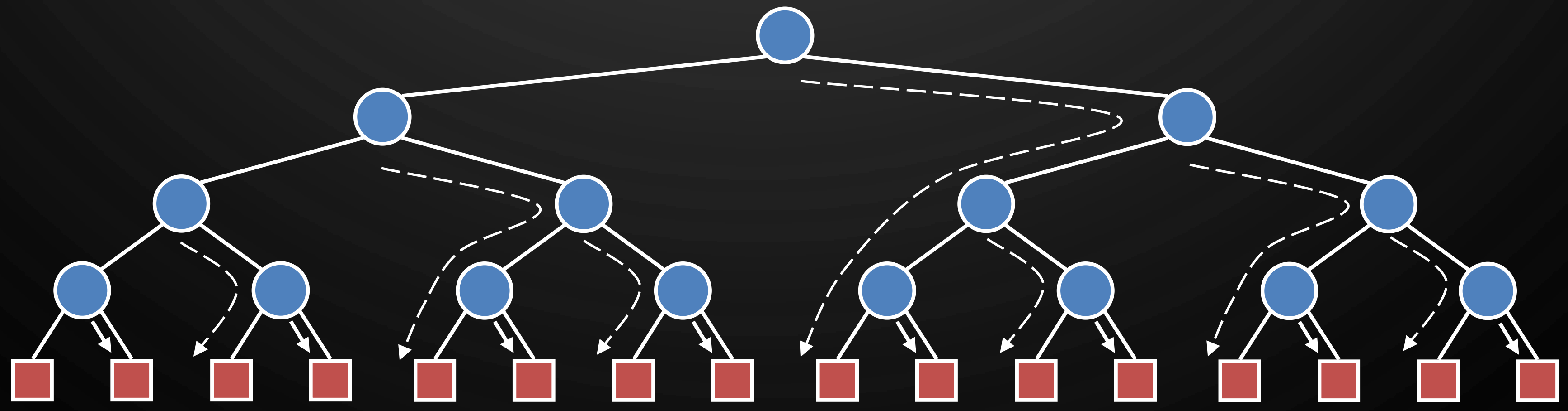


EXAMPLE



ANALYSIS

- We visualize the worst-case time of a downheap with a proxy path that goes first right and then repeatedly goes left until the bottom of the heap (this path may differ from the actual downheap path)
- Since each node is traversed by at most two proxy paths, the total number of nodes of the proxy paths is $O(n)$
- Thus, bottom-up heap construction runs in $O(n)$ time
- Bottom-up heap construction is faster than n successive insertions and speeds up the first phase of heap-sort

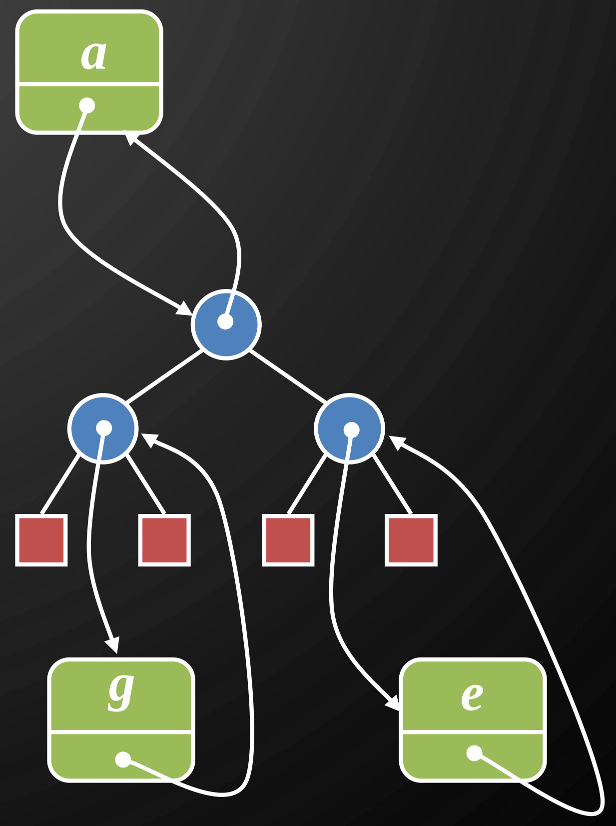


ADAPTABLE PRIORITY QUEUES

- One weakness of the priority queues so far is that we do not have an ability to update individual entries, like in a changing price market or bidding service
- We incorporate concept of positions to accomplish this (similar to List)
- Additional ADT support (also includes standard priority queue functionality)
  - $\text{insert}(e)$ – insert element e into priority queue and return a position referring to this entry
  - $\text{remove}(p)$ – remove the entry referenced by position p
  - $\text{replace}(p, e)$ – replace with e the element associated with position p and return the position of the altered entry

LOCATION-AWARE ENTRY

- Locators decouple positions and entries in order to support efficient adaptable priority queue implementations (i.e., in a heap)
- Each position has an associated locator
- Each locator stores a pointer to its position and memory for the entry



POSITIONS VS. LOCATORS

- Position
  - represents a “place” in a data structure
  - related to other positions in the data structure (e.g., previous/next or parent/child)
  - often implemented as a pointer to a node or the index of an array cell
- Position-based ADTs (e.g., sequence and tree) are fundamental data storage schemes

- Locator
  - identifies and tracks a (key, element) item
  - unrelated to other locators in the data structure
  - often implemented as an object storing the item and its position in the underlying structure
- Key-based ADTs (e.g., priority queue) can be augmented with locator-based methods